José Luis Escamilla Reyes
(organizador)

EDUCAÇÃO E ENSINO DE CIÊNCIAS EXATAS E NATURAIS

VOL II

EDITORA ARTEMIS
2024

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Bibliotecário	Maurício Amormino Júnior – CRB6/2422

Editora Artemis
Curitiba-PR Brasil
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Dados Internacionais de Catalogação na Publicação (CIP)
(eDOC BRASIL, Belo Horizonte/MG)

E24 Educação e ensino de ciências exatas e naturais II [livro eletrônico] / Organizador José Luis Escamilla Reyes. – Curitiba, PR: Artemis, 2024.

Formato: PDF
Requisitos de sistema: Adobe Acrobat Reader
Modo de acesso: World Wide Web
Inclui bibliografia
Edição bilíngue
ISBN 978-65-81701-29-1
DOI 10.37572/EdArt_311024291

1. Educação. 2. Ciências exatas e naturais – Estudo e ensino. 3. Professores – Formação. I. Reyes, José Luis Escamilla.

CDD 371.72

Elaborado por Maurício Amormino Júnior – CRB6/2422

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Editora Artemis
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PRÓLOGO

En este volumen, se presentan los resultados de varios y diversos proyectos de investigación en innovación educativa relacionados con la enseñanza de las ciencias y la ingeniería, tanto en niveles universitarios como básicos. Es así como, a través de distintas experiencias, se aborda la enseñanza de la Física, la Química Analítica y la enseñanza de temas matemáticos tales como la Aritmética y el Álgebra. También, se explora la incorporación de nuevas alternativas como la Inteligencia Artificial y sus aplicaciones en la enseñanza de las ciencias, particularmente de la Química.

Adicionalmente, en este libro se discuten los procesos de evaluación, no sólo de las actividades realizadas por los alumnos en los diferentes niveles educativos, sino de la pertinencia y adecuación del currículum en las disciplinas científicas, dentro de las que se puede mencionar a la Química Analítica y las Ciencias Exactas en general.

Por supuesto, hago la invitación a nuestros lectores para que disfruten la lectura de estos artículos de innovación educativa y, si son docentes en activo, que implementen alguna o varias de las estrategias y metodologías expuestas en este volumen con el fin de enriquecer su práctica docente y, de esta manera, contribuir en la mejora de los procesos educativos desde los niveles básicos hasta los universitarios.

Finalmente, los autores de este libro agradeceremos la retroalimentación y los comentarios propositivos que nos hagan llegar, puesto que lo más importante es asegurar que nuestros alumnos tengan una educación de calidad y que logren un aprendizaje significativo que les permita superar con éxito los problemas tanto en su formación académica como en su vida cotidiana.

Dr. José Luis Escamilla Reyes

SUMÁRIO

CAPÍTULO 9

DEVELOPING LEARNERS' ALGEBRAIC MANIPULATION ABILITY: A MATHEMATICS TEACHER EDUCATOR REFLECTS ON PRE-SERVICE TEACHERS' INITIAL THOUGHTS

Data de submissão: 15/09/2024
Data de aceite: 02/10/2024

Barbara Kinach, Ed.D.
Arizona State University
Tempe, Arizona, USA
https://orcid.org/0000-0003-0253-9213

ABSTRACT*:* This design research study applies the cognitive strategy IACTS (Identify Assess Challenge Transform Sustain) previously developed for integer subtraction and multiple meanings of the "–" symbol to the context of symbolic transformation of algebraic expressions involving the "–" symbol. Qualitative data for this exploratory study conducted during Fall 2022 in the author's secondary mathematics methods course include preservice teachers' written explanations for how to teach –(x+y), 2 – (x + 3), and 2 – (x – 5) "from scratch" and their post-course reflections on this experience. Preliminary results reported by the mathematics teacher educator indicate PSTs' understanding of subtraction as adding the opposite did not fully transfer initially from the integer arithmetic to the algebraic context thus creating a need for PCK-development related to methods of teaching simplifying algebraic expressions involving the "–" symbol.
KEYWORDS: PCK for Simplifying Algebraic Expressions.

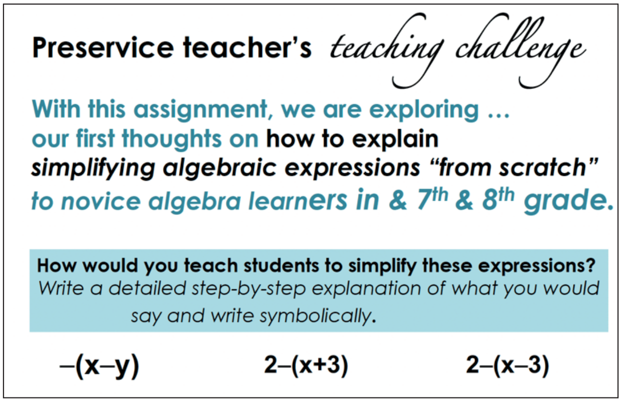
1 INTRODUCTION

Passion by one definition is a strong enthusiasm or desire for anything (as in a passion for music). It has been my passion as a mathematics teacher educator for the past three decades to develop preservice teachers' passion and motivation to teach mathematics conceptually for all learners. Accomplishing this goal in the secondary mathematics methods courses that I teach can be challenging as often my secondary mathematics preservice teachers (PSTs) are very confident in their current understanding of mathematics and may not appreciate the conceptual hurdles novice learners confront in learning complex mathematics topics such as algebraic manipulation. Thus, as their mathematics methods-course instructor, I cannot launch our study of mathematics learning and teaching by simply saying, "In this course, I will be asking you to teach algebra in reasoned ways that are perhaps different from the ones you've likely experienced and learned in your own education." A bit more nuanced of an approach is required to first spark pre-service teachers' perception of the need for conceptual understanding in algebra, and subsequently build both their inner motivation to teach algebra conceptually and their belief in the ability to teach algebra meaningfully in schools where students and teachers immersed in the direct instruction approach foster the learning and teaching of algebra through memorized tricks and rules without reasoned mathematical support. This study shares one exemplar of mathematics teacher educator practice in the secondary mathematics methods course that I teach (hereafter called "methods") that aims to stimulate and develop novice teachers' passion for, understanding of the need for, skill, and belief in their ability to teach mathematics, and specifically algebraic manipulation when simplifying algebraic expressions, in a reasoned way that promotes conceptual (instead of mindless procedural) understanding of algebraic expressions.

2. PURPOSE OF THE STUDY

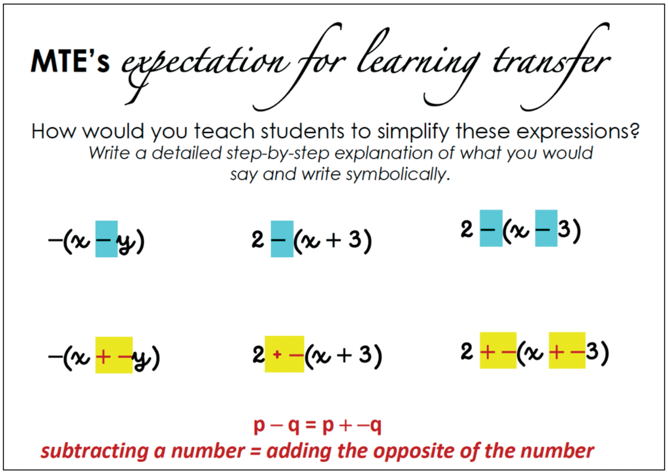
The purpose of the study was to investigate the effectiveness of a hypothesized learning trajectory (LT) (Simon, 1993) for the development of prospective teachers' PCK related to simplifying algebraic expressions through algebraic manipulation. The proposed LT began with a unit on integer subtraction and interpreting the meaning of the "–" symbol according to the syntax of numerical expressions involving integers and integer operations. Re-contextualizing concepts learned within the integer unit for the algebraic context, the mathematics teacher educator (MTE) hypothesized a learning sequence designed for use in the methods course to guide pre-service teachers in learning-to-teach simplifying algebraic expressions involving the "–" symbol. To launch prospective teachers' thinking about how they would teach middle grades 6 through 9 algebra students to simplify expressions involving the "–" symbol, the MTE posed a prompt containing three algebraic expressions drawn from the literature on the PCK of expressions and equations (De Araujo, Dougherty & Zenigami, 2018). Specifically, the prompt asked prospective middle and early secondary mathematics teachers to reflect on the question: *How would you teach students to simplify these expressions: –(x–y), 2–(x+3), 2–(x–3)? Write a detailed step-by-step explanation of what you would say and write symbolically* (Figure 1).

Figure 1. Study Prompt.



The MTE hypothesized that prospective teachers' prior reasoning about syntax and the multiple meanings of the "–" symbol (i. e., subtract, negative or opposite) within the integer subtraction unit previously completed within the methods course would transfer to reasoning syntactically about the multiple meanings of the "–" symbol when learning-to-teach algebraic manipulation when simplifying algebraic expressions. Specifically, for the three algebraic expressions examined in this study, the MTE anticipated that in thinking about how to teach simplifying algebraic expressions from scratch, PSTs would envision teaching their novice algebra learners to apply the definition of integer subtraction previously learned from the (National Governors Council, 2010) State Standards for Mathematics during a unit on integer operations to each subtraction symbol in the study's three algebraic expressions. To apply the definition of integer subtraction as adding the opposite (e.g., $p - q = p + {}^{-}q$), the MTE anticipated PSTs would encourage their students to change each of the subtraction symbols in an algebraic expression to "adding the opposite" and subsequently lead a whole class discussion to help their students syntactically determine the meaning of the remaining "–" symbols in the expressions as either negative or opposite (Figure 2). The overall aim was to 1) determine whether transfer of the multiple meanings of the "–" symbol occur naturally when moving from learning to teach the "–" concept within the integer-arithmetic context to learning to teach the "–" concept within the algebraic context and 2) perfect the design of an effective LT for the methods course for developing PSTs' mathematics content and pedagogical content knowledge related to symbolic transformation of algebraic expressions involving the "–" symbol prior to learning to teach the equation concept.

Figure 2. MTE Expects PSTs to Teach Simplifying Algebraic Expressions by Applying the Definition of Subtraction as "Adding the Opposite" to all Subtraction Symbols in Algebraic Expressions.



3. PERSPECTIVES AND THEORETICAL BACKGROUND

As the study focuses on the development of pedagogical content knowledge for pre-service teachers, a review of the relevant PCK literature related to simplifying algebraic expressions involving the "–" symbol is warranted.

Pedagogical content knowledge (PCK) is the understanding of content, learners, pedagogy, and curriculum that mathematics teacher educators foster in teacher preparation programs to encourage pre-service teachers to teach mathematics conceptually instead of by memorizing (Shulman, 1986). In teacher preparation programs, the methods course is the primary site for development of prospective mathematics teachers' PCK. Kinach (2002) proposed a cognitive strategy for developing prospective teachers' PCK in the methods course. The five-part cognitive strategy, called IACTS, aims to Identify, Assess, Challenge, Transform, and Sustain conceptual changes in the procedural understanding preservice teachers' often hold about the mathematics they are preparing to teach by focusing on prospective teachers' notions of a "good" instructional explanation. For example, by applying the IACTS cognitive strategy to investigate PSTs' understanding of integer subtraction, the prior study by Kinach (2002) identified common misconceptions and lack of awareness among pre-service secondary mathematics teachers about the multiple meanings of the "–" symbol. Results demonstrated pre-service teachers' computation practices for integer addition and subtraction to be largely procedural based on tricks such as "minus a minus is a plus." The current study builds on these prior findings.

The cognitive strategy Kinach (2002) proposed for unearthing prospective secondary teachers' procedural understanding of the mathematics content they are preparing to teach begins by investigating prospective teachers' untutored ideas about "good" instructional explanations for a given topic. The approach involves 5 phases: I:Identify, A:Assess, C:Challenge, T:Transform, S:Sustain. For example, the first step in applying IACTS to build strong mathematics content and pedagogical content knowledge for prospective teachers for a specific concept (e.g., integer subtraction) involves asking pre-service teachers to explain the targeted concept or procedure (e.g., integer subtraction 5 - -3) in whatever way comes naturally. This is the I:Identify phase.

During the next A:Assess phase, the adequacy of the instructional explanations is determined in light of current depth-of-knowledge frameworks (e.g., Perkins & Simmons, 1988) and standards directives (NGA & CCSO, 2010). Do, for example, the proposed instructional explanations foster thinking at the concept, problem-solving, and justification levels of understanding? Do the instructional explanations foster analysis, contextualization, production of examples and patterns for generalization? The answers to questions such as these reveal the need for revision (or not).

During the next C:Challenge phase Socratic questioning by the methods instructor may challenge prospective teachers' notions of a "good" instructional explanation. Are the proposed instructional explanations likely to lead to meaningful understanding of the concept? Do they promote depth of knowledge or only superficial understanding of information and skills?

During the T:Transform phase, prospective teachers begin to transform their instructional explanations into forms more likely to lead to conceptual understanding. To do this, the methods instructor quietly orchestrates a step backwards. To create a bit of cognitive conflict, the methods instructor asks PSTs to explain X again but this time in a context A less likely to lead to easy representation of the mathematical concept or process. By holding any logical tension that develops through Socratic questioning, the methods instructor sows doubt about the adequacy of the instructional explanations until prospective teachers are convinced the cognitive conflict must be resolved and changes made to the instructional explanations to make the concept more accessible for their future student.

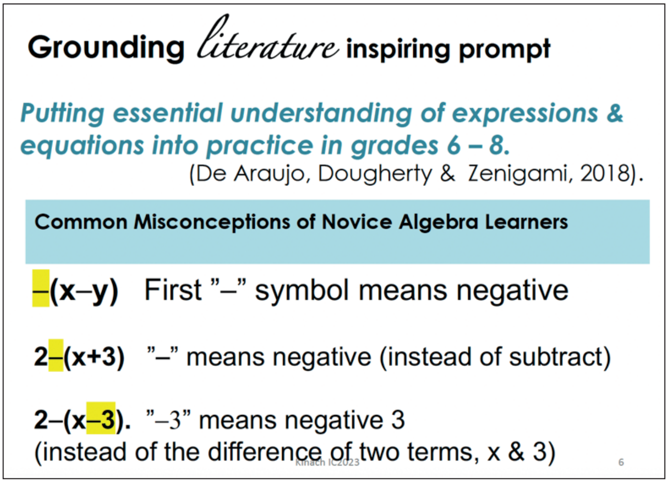
During the S:Sustain phase the methods instructor asks prospective teachers to explain the concept again but this time in a new context B that lends itself to clear and precise representation of the targeted concept. Using Socratic questioning again, the methods instructor encourages PSTs to use the Depth of Knowledge frameworks explored during the A:Assess phase to determine the adequacy of their initial explanation along with the explanations produced in contexts A and B. Inevitably, the three explanations are fuel for debate over the meaning and the most effective way to represent and explain the concept to novice learners.

Documenting the mathematics content knowledge for expressions and equations that teachers require, Lloyd, Eisenmann, and Star (2011) detail the essential understanding of symbolic transformations teachers need to know to teach students how to generate equivalent expressions algebraically through symbolic transformations. Among the transformations cited by Lloyd et al. applicable to this study are the distributive property of multiplication over addition, expressed for all *R*eal numbers as $a(b + c) = ab + ac$, and the opposite of a sum property, which asserts that *the opposite of a sum equals the sum of the opposites* which is represented symbolically as $-(a + b) = -a + -b$.

De Araujo, Dougherty, and Zenigami (2018) put the essential ideas laid out by Lloyd et al. into practice for teachers of grades 6-8. These researchers aim to articulate the PCK that prospective and all teachers require to represent and explain expressions and equations in meaningful ways for the population of students in grades 6 through 8. De Araujo et al. elaborate the typical misconceptions and error patterns that middle grades 6-8 students often display while learning to simplify algebraic expressions when reasoning with mathematical definitions and symbolic transformations to generate equivalent algebraic expressions. As will be detailed later in this study, several of the misconceptions involving the "–" symbol that middle grades and early secondary students display are identical to the ones my pre-service teachers displayed as we worked on developing PCK for teaching simplifying algebraic expressions through algebraic manipulation.

The De Araujo et al. research inspired the current study. The algebraic expressions –(x–y), 2–(x+3). and 2–(x–3) reported by De Araujo, Dougherty, and Zenigami in their study of the misconceptions and error patterns of early algebra learners are the focus of this study. The examples of learner misconceptions reported by De Araujo et al. include consistent misinterpretation of the "–" symbol as negative regardless of syntax. Specifically, for example, for the expression –(x–y), students in grades 6-8 misinterpret the first "–" symbol as negative (instead of "opposite") while for the expressions 2–(x+3) and 2–(x–3) middle grades students interpret all of the "–" symbols as negative (instead of "subtract"). These examples of misinterpretation of the "–" symbol extend the prior findings of Kinach (2002) regarding misinterpretation of the "–" symbol in the integer arithmetic context to the algebraic context for expressions involving the "–" symbol (Figure 3).

Figure 3. Research on Algebra Learner Misconceptions about the "–" Symbol.



4. PARTICIPANTS, CONTEXT, DATA COLLECTION, ANALYSIS, METHODS

4.1 PARTICIPANTS & CONTEXT

Participants are a convenience sample of the 17 undergraduate pre-service teachers (13 females and 5 males) enrolled in the author's Fall 2022 methods course, *Methods of Teaching Standards-Based Middle Grades and Early Secondary Mathematics.* Qualitative data for this exploratory design study include pre-service teachers' written explanations for how to teach novice algebra learners to simplify the algebraic expressions –(x + y), 2 – (x + 3), and 2 – (x – 5) "from scratch" along with their post-course reflections on this experience.

4.2 DATA COLLECTION & ANALYSIS

Data was collected in class. PSTs each completed three google slides, one for each of the three algebraic expressions, by responding to the previously cited prompt (Figure 1) inviting them to explore first thoughts on how they would teach students to simplify each of the three expressions: –(x + y), 2 – (x + 3), and 2 – (x – 5). The prompt, which was given with the expectation that PSTs would teach their students to simplify the algebraic expressions by applying the previously learned definition of subtraction by changing each subtraction symbol in the algebraic expressions to "adding the opposite" and then reasoning syntactically to determine the meaning of the remaining "–" symbols as negative or opposite, yielded unexpected results.

Data analysis consisted of a review of the 51 google slides and enumeration of PSTs' approaches to simplifying and interpreting the "–" symbol within each of the three algebraic expressions. PST approaches are reported later in the study along with the mathematics teacher educators' reflection on PSTs' approaches to simplifying algebraic expressions involving the "–" symbol. The MTE's notes on a hypothetical learning trajectory for developing pre-service teachers' PCK on simplifying algebraic expressions completes data collection for the study.

4.3 PCK DEVELOPMENT TEACHING METHOD

As previously described, the PCK development teaching method called IACTS (Identify Assess Challenge Transform Sustain) is a cognitive strategy Kinach (2002) proposed for probing, assessing, and if necessary, transforming prospective teachers' instructional explanations from a procedural to a more meaningful conceptual understanding.

In this design research study, the MTE applied two cycles of the IACTS cognitive strategy to develop PSTs' PCK for simplifying algebraic expressions. During the first cycle, the MTE applied IACTS to develop PCK for integer subtraction and the multiple meanings of the "–" symbol within the integer subtraction context. During the second cycle, the MTE applied IACTS to develop PCK for simplifying algebraic expressions involving the "–" symbol based on the multiple meanings of the "–" symbol learned during the first IACTS cycle for integers.

Additionally, for this study, there was a unique overlap of the two cycles in that the end of the first IACTS (integer subtraction) cycle served as the beginning of the second IACTS (simplifying algebraic expressions) cycle . Specifically, after guiding PSTs through the first four phases of the (integer subtraction) IACTS cycle (I:Identify, A:Assess, C:Challenge, and T:Transform) to establish PST ability to determine the meaning of the "–" symbol syntactically for numerical integer expressions, the MTE simultaneously launched the S:Sustain cycle for integer subtraction and the I:Identify cycle for simplifying algebraic expressions involving the "–" symbol. One activity served both purposes. By asking pre-service teachers to do what comes naturally to simplify the three algebraic expressions: $-(x+y)$, $2 - (x + 3)$, and $2 - (x - 5)$, the MTE determined whether PSTs could S:Sustain their understanding of the syntax of the "–" symbol for integers in the new algebraic context while also I:Identifying PSTs' starting point for reasoning about how to teach simplifying algebraic expressions involving the "–" symbol to novice algebra learners.

Continuing the second IACTS cycle of PCK development for the algebraic context, the MTE then chose to A:Assess the adequacy of PSTs' instructional explanations for simplifying each of the three algebraic expressions against the criterion of syntactic knowledge about the multiple meanings of the "–" symbol learned during the integer IACTS cycle and context.

As it was necessary, the MTE then chose to C:Challenge prospective teachers' notion of "good" ways to explain and teach simplifying algebraic expressions involving the "–" symbol by asking PSTs whether their method was likely to lead to meaningful understanding of how to reason with and apply mathematical definitions and principles in order to manipulate algebraic symbols when simplifying algebraic expressions or whether their method was likely to yield mathematically unsupported memorized rules and tricks for simplifying algebraic expressions involving the"–" symbol. The C:Challenge discussion is an opportunity for PSTs to create a carefully sequenced accessible learning path for novice algebra students who are learning to simplify algebraic expressions involving the "–" symbol for the first time.

During the T: Transform phase of the (simplifying algebraic expressions) IACTS cycle, PSTs create ways for middle schoolers to identify the syntax of the "–" symbol. When does the "–" symbol mean subtract? When does the "–" symbol mean negative? When does the "–" symbol mean opposite? Preservice teachers will need to solidify their ideas on this point before finalizing their views on "good" instructional explanations for teaching "simplifying algebraic expressions involving the "–" symbol.".

To close the IACTS cycle of PCK development for simplifying algebraic expressions, the S:Sustain cycle asks PSTs to re-consider their initial approach by converting all subtraction symbols for the study's three algebraic expressions to "adding-the-opposite" and then reason syntactically to determine the meaning of the remaining "–" symbols as either negative or opposite. Once accomplished, a final "clinching" algebraic expression is posed for prospective teachers to try out their simplified pedagogy for teaching students in grades 6-9 to simplify algebraic expression involving multiple meanings of the "–" symbol.

5 RESULTS/FINDINGS

How would you teach students to simplify these expressions? Write a detailed step-by-step explanation of what you would say and write symbolically

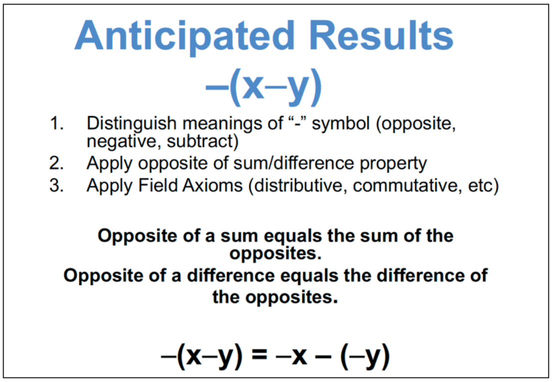
1. $-(x - y)$
2. $2 - (x + 3)$
3. $2 - (x - 3)$

Interpretation of the "–" symbol in PSTs' responses to the above prompt varied widely both within and across the three expressions. PSTs' responses differed from the mathematics teacher educator's expectation that they would apply the multiple meanings of the "–" symbol learned in the integer context to the algebraic setting. Specifically, the mathematics teacher educator (MTE) expected PSTs to change every subtraction symbol in the three prompt expressions to adding the opposite following the definition of rational number subtraction in the Common Core State Standards for Mathematics (NGA & CCSO, 2010), specifically, $p - q = p + {}^{-}q$ (Figure 2). Brief analyses of PSTs' approaches to simplifying the three expressions listed above follows.

5.1 INTERPRETATION OF THE "–" SYMBOL FOR THE EXPRESSION – (X – Y)

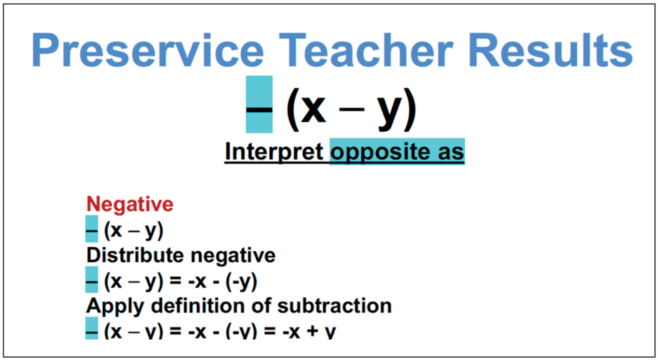
The MTE anticipated PSTs would apply the "opposite of a sum" transformation property that states: *The opposite of a sum equals the sum of the opposites, or* $-(a + b) = -a + -b$. (Figure 4).

Figure 4. MTE Expects PSTs to Apply "Opposite of a Sum" Transformation Property to –(*a* + *b*).



For the expression – (x – y), PSTs misinterpreted the "–" symbol outside the parentheses as either negative or subtract, but not opposite (Figure 5).

Figure 5. PST Misinterpret the "Opposite" Symbol as "Negative" or "Subtract".

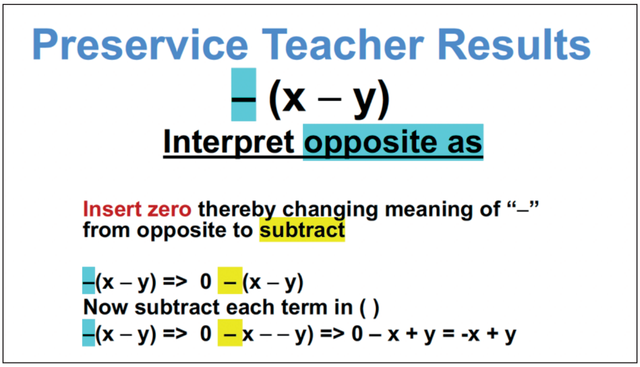


When misinterpreted as a negative sign, PSTs incorrectly talked about "distributing the negative" to the parentheses. This was done in two ways. Either the "negative sign" was distributed to the x and the misinterpreted "negative y" in the parentheses, or the "negative sign" was distributed to the minuend and subtrahend, yielding -x - -y or -x + y (Figure 5).

When misinterpreted as a subtraction symbol, PSTs reasoned about subtracting the x and (-y) to yield -x – (-y). While not incorrect mathematically, the symbol outside the parentheses does not indicate subtraction as the binary operation of subtraction requires two elements (minuend and subtrahend) to execute the operation.

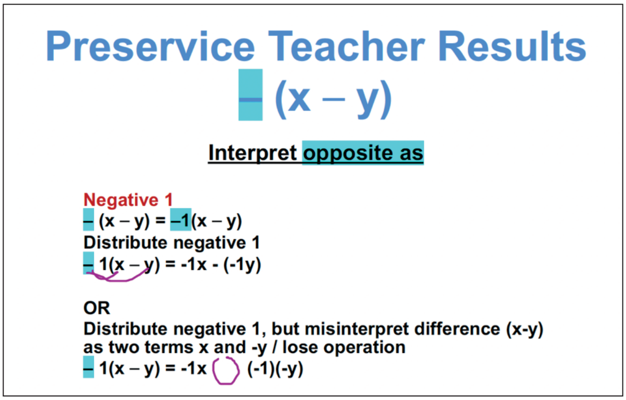
Another approach to simplifying – (x – y), inserts a zero before the expression – (x – y) to create 0 – (x – y). This approach inadvertently changes the opposite symbol to subtraction without mathematical reason (Figure 6).

Figure 6. Inserting Zero Before Opposite Symbol Changes Symbol Meaning to "Subtract".



Yet another approach PSTs proposed to simplify the expression – (x – y) involves inserting "negative one" just outside the parentheses hich inadvertently changes the meaning of the "– " symbol from "opposite" to "negative." PSTs proceed by multiplying the terms inside the parentheses by negative one (-1). Difficulties of interpretation additionally arise inside the parentheses when the subtrahend is interpreted correctly as y or incorrectly as (–y) (Figure 7).

Figure 7. PSTs Change Meaning of “Opposite” Sign to “Negative” by Inserting “1” before the Parentheses.

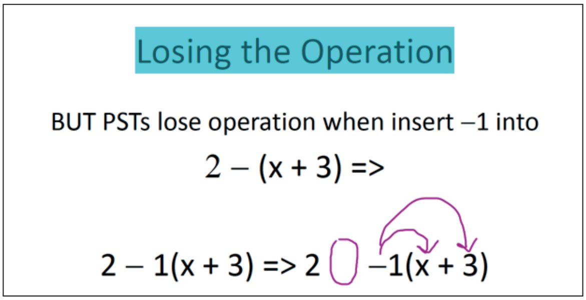


5.2 INTERPRETATION OF THE “–” SYMBOL FOR THE EXPRESSION 2 – (X + 3)

For the second expression 2 – (x + 3) and its counterpart, 2 – (x – 3), similar issues arise concerning interpretation of the first “–” symbol as negative. But when correctly interpreted as “subtract” PSTs wither subtract both minuend and subtrahend separately (2 – x – 3) or apply the definition of subtraction as adding the opposite: $2 + {}^{-}(x + 3) = 2 + {}^{-}x + {}^{-}3$. The latter interpretation is the one expected by the MTE based on prior methods course discussions.

Another approach to simplifying 2 – (x + 3) involved inserting a “1” before the parentheses. Inserting the “1” does not change the meaning of the **“–”**symbol as there is a term before and after the **“–”** symbol indicating **“–”** to be a binary operation. As soon as PSTs insert the 1, however, they interpret the **“–”** symbol as “negative” and distribute the “negative one” to the parentheses. In so doing, PSTs “lose” the operation between the terms “2” and (x+3), making it unclear how the two terms are to be related operationally (Figure 8).

Figure 8. PSTs “Lose” Operation by Interpreting the 1 as “Negative One”.

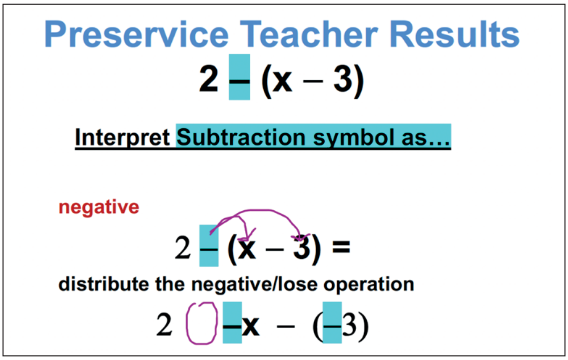


5.3 INTERPRETATION OF THE "–" SYMBOL FOR THE EXPRESSION 2 – (X – 3)

For the expression 2 – (x – 3), PSTs interpreted the first "–" symbol in three ways: negative, negative one, and remove-the-bracket rule. At no time did PSTs interpret the first "–" symbol as the operation of subtraction.

Figure 9 displays the algebraic manipulation linked to interpretation of the first "–" symbol as negative. PSTs display a misunderstanding of the transformation property "distributive property of multiplication over addition/subtraction" by distributing the "negative" to each term in the parentheses. Notice how the operation between the two terms "2" and (x–3) is lost (Figure 9) when the subtraction symbol is misinterpreted as negative.

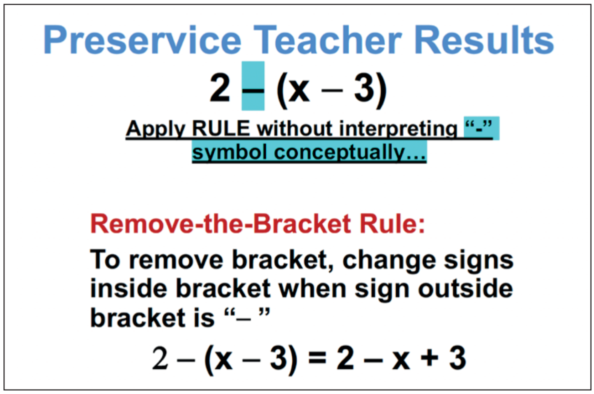
Figure 9. PST "Distribute" the "Negative" thereby Misapplying the Distributive Property.



In Figure 9, misunderstanding of the distributive property of multiplication over subtraction, *a* x (*b* + *c*) = *a* x *b* + *a* x *c,* for all real numbers is evident. PSTs need to recognize that the element being distributed when using the distributive property to transform an expression into an equivalent one, must be a *R*eal number (and not a negative symbol). Although the procedure used by PSTs results in an accurate mathematical expression this approach is not one recommended to teach to novice algebra learners tackling algebraic manipulation for the first time. Note the reason the procedure works is because it essentially converts subtracting "2" and (x–3) into adding the opposite of (x–3) to 2, which is the formal definition of rational number subtraction.

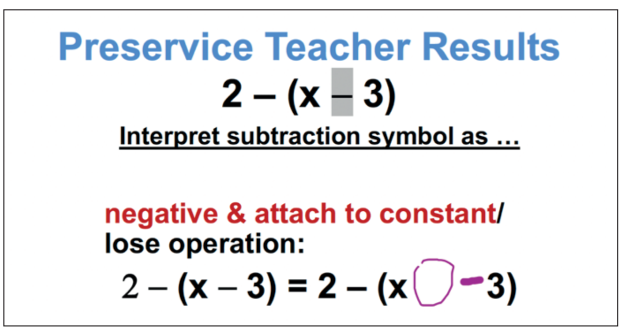
Another approach to simplifying 2 – (x–3) was the "remove the bracket" rule (Figure 10). This is not a recognized transformation property according to Lloyd, Eisenmann, and Star (2011). The rule proposed by the pre-service teacher was "to remove the bracket, and then change the signs in the parentheses;" for example 2 – (x – 3) becomes 2 – x + 3.

Figure 10. MTE Expectation PSTs Would Apply Integer.



PSTs interpreted the second "–" symbol in the parentheses for 2 – (x – 3) in two ways. On the one hand, they interpreted the second "–" symbol correctly as subtraction, but on the other hand they misinterpret the second "–" symbol as negative when they attach the "–" symbol to the subtrahend thereby changing "x minus positive three" into "x" and "negative three" without an operation between them (Figure 11).

Figure 11. PSTs Attach Subtraction Symbol to Subtrahend and "Lose" the Operation.



6 QUANTITATIVE RESULTS & DISCUSSION

As the table in Figure 12 shows, PSTs interpreted the "–" symbol correctly as subtraction in only 26 of 68 cases, or about thirty-eight percent (26/68= .382) of the time. By contrast, PSTs misinterpreted the subtraction symbol as negative 54% (37/68 = .544) of the time. On two occasions (in about 3% of cases or 2/68 =.029) the subtraction symbol was interpreted as "opposite" while in 7 instances (about 10% of cases or 7/68=.102) the subtraction symbol triggered a procedure that was not supported by any mathematical rationale or principle. Additionally, on two occasions two students interpreted a single "–" symbol in two ways (as both negative and subtract).

Figure 12. PST Accurate & Inaccurate Interpretation of the Subtraction Symbol.

PSTs' Accurate Interpretation of "-" Symbol as Subtract: Results Overview

	-(x-y)	-(x-y)	2-(x+3)	2-(x-3)	2-(x-3)
Subtract 26/68 = 38%		7	6	7	6
Negative 37/68 = 54%		8	11	9	9
Opposite 2/68 = 3%		0	1	1	0
Rule w/o Concept 7/68 = 10%		2	1	2	2
		17	19•	19•	17
* 2 students choose both negative & subtract					

The table in Figure 13 shows PST success with interpreting the meaning of the "–" symbol in situations where it means "opposite." PST interpreted the "–" symbol accurately as "opposite" in only two of 17 total occasions. Thus, PSTs accurately interpreted the "opposite" symbol only 12% of the time. By contrast, PSTs mostly interpreted the "opposite" symbol to mean "negative." This occurred in 12 of 17 instances, or approximately 71% (12/17 = .705) of the time. Additionally, on 3 occasions, PSTs inaccurately interpreted the "opposite" symbol to mean "subtract" which is about 18% (3/17 = .176) of the time.

Figure 13. PST Accurate & Inaccurate Interpretation of the Opposite Symbol.

PSTs' Accurate Interpretation of "-" Symbol as Opposite: Results Overview

	-(x-y)	-(x-y)	2-(x+3)	2-(x-3)	2-(x-3)
Subtract 3/17 = 18%	3				
Negative 12/17 = 70%	12				
Opposite 2/17 = 12%	2				
Rule w/o Concept 0/17 = 0%	0				
	17				

In this class of 17 pre-service mathematics teachers, what was most surprising to the MTE is that very few transferred prior understanding of the syntax of a subtraction symbol from the integer to algebraic context. If PSTs had recognized the syntax in the new algebraic setting, they would have applied the definition of subtraction learned for integers ($p - q = p + {}^{-}q$) to the expressions $2 - (x + 3)$ and $2 - (x - 3)$ (Figure 1). Mostly, PSTs did not interpret the "–" symbol as subtraction but as a negative sign or invisible "negative one."

And while each PST approach interestingly yielded an accurate mathematical result, as we have seen in the prior section, not all approaches were motivated by a legitimate mathematical principle. "Distributing the negative," for example, was a misconception that required clarification of the meaning of the distributive property of multiplication over addition for all *R*seal numbers. Since the distributive property $a(b + c) = ab + ac$ involves distributing the *R*eal number (a), it is technically incorrect to think of $-(a + b)$ as an opportunity to "distribute" the symbol outside the parentheses as it is not a *R*eal number. Most PSTs interpreted the *"–" symbol* outside the parentheses as meaning "negative" and then transformed $-(a + b)$ to an equivalent expression by "distributing the negative" which technically is not a legitimate mathematical action/principle. Applying the transformation property called "opposite of a sum" is the more mathematically accurate way to reason about transforming $-(a + b)$ to its equivalent expression, ${}^{-}a + {}^{-}b$. As a result of the 'simplifying algebraic expressions" activity, PSTs learned what (surprisingly) turned out to be for them a transformation property about which they were previously unaware, namely that: *the opposite of a sum* equals *the sum of the opposites,* or symbolically represented as $-(a + b) = {}^{-}a + {}^{-}b$.

7 CONCLUSION: MATHEMATICS TEACHER EDUCATOR REFLECTION

PSTs' responses were surprising to the methods instructor, the mathematics teacher educator who is also the author of this paper. For the most part, extensive prior discussion of the multiple meanings of the "–" symbol in the integer subtraction context did not transfer to reasoning about the "–" symbol in the algebraic context. Why did this occur? What changes might prevent this from happening? Do we *want* to prevent this from happening? Or are the insights gained from the multiple approaches to simplifying the three algebraic expressions involving the "–" symbol more instructive for prospective algebra teachers and their mathematics methods instructor? Are the debates over method, over misinterpretations of the "–" symbol, and missing information about transformation properties an integral part of the emerging PCK information library and learning process for "simplifying algebraic expressions involving the "–" symbol"?

The variety of meanings and approaches PSTs proffered for the "–" symbol is interesting to think about from the perspective of the task. The purpose of the task was to create a pedagogy suitable for teaching novice algebra learners to *simplify algebraic expressions involving the "–" symbol* for the very first time. When one thinks about the mission of algebra teachers whose charge it is to induct young algebraists into the syntax of algebraic manipulation through precise reasoning and skillful application of mathematical definitions, principles, and symbolic transformations, one is led to consider the advisability of teaching a range of approaches at the beginning of a unit on algebraic manipulation, equivalent expressions, and transformation properties. This is a conversation that the methods instructor had with her prospective algebra teachers. Without taking a formal count, the class generally agreed that early instruction related to simplifying algebraic expressions ought to be "reasoned" and governed by application of accessible mathematical definitions, principles, and transformation properties for the middle/early secondary grade population of students.

Imagining these pre-service teachers instructing their future students 17 different ways to solve each of the three expressions is a daunting vision, especially as the reasoning underlying the manipulation of the algebraic symbols was often mathematically inaccurate albeit computationally successful. The debates that the PSTs had with their MTE over the reasoning employed to simplify these expressions was rich, eye-opening motivation for everyone to be more conceptually precise. Next steps will involve duplicating the design experiment, expanding complexity of algebraic expressions involving the "–" symbol, and developing a methods-course tested learning trajectory and instructional sequence for teaching pre-service teachers to teach simplification of algebraic expressions "from scratch" through reasoned application of mathematical definitions, transformation properties, and mathematically accurate algebraic manipulation.

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SOBRE O ORGANIZADOR

Dr. José Luis Escamilla Reyes- Profesor del Tecnológico de Monterrey, Campus Ciudad de México desde 1998. Doctor en Física por la Universidad Autónoma Metropolitana, Unidad Iztapalapa. Cuenta con una experiencia docente de 32 años. Es coautor de Manuales de Física II y Física III, así como de dos ebooks, uno sobre Física General y otro sobre Óptica y Física Moderna. Está certificado en el Programa de Desarrollo de Habilidades Docentes del Tecnológico de Monterrey. Ha participado con varios trabajos en Congresos Nacionales e Internacionales relacionados con la Física de Semiconductores de los grupos IV y III-V. Sus áreas de interés son: fuentes alternativas de energía, Física del Estado Sólido, diseño y aplicaciones de los MEMS y modelación matemática de Sistemas Complejos. Ha publicado más de 15 trabajos arbitrados y memorias en congresos. Colaboró en el diseño y construcción de láseres pulsados de N2 en el Laboratorio de Óptica Cuántica de la Universidad Autónoma Metropolitana Iztapalapa (UAMI). En el Tecnológico de Monterrey Campus Ciudad de México, participó en el desarrollo de un prototipo de Celda de Combustible con membrana de intercambio protónico (PEMFC) de alta eficiencia. Obtuvo la Medalla al Mérito Académico por el mejor promedio de Maestría otorgada por la UAMI. Fue líder de la Cátedra de Investigación “Micro Sistemas Electromecánicos: Diseño y aplicaciones” del Tecnológico de Monterrey, Campus Ciudad de México y miembro del SNI.

ÍNDICE REMISSIVO

A

C

D

E

F

I