VOL III

Educação:

Saberes em Movimento, Saberes que Movimentam

Teresa Margarida Loureiro Cardoso

(organizadora)

EDITORA ARTEMIS

2023

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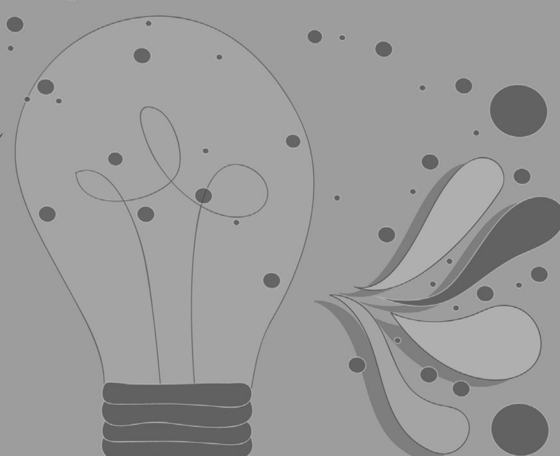

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APRESENTAÇÃO

O terceiro volume da *Educação: Saberes em Movimento, Saberes que Movimentam*, publicado pela Editora Artemis, convoca a inovação enquanto denominador comum. Uma inovação em torno de diversos cenários digitais, que hoje, mais do que nunca, populam os nossos quotidianos, em diferentes níveis de ensino. Mas também uma inovação em torno de competências de literacia ditas básicas, tradicionais, como a leitura e a escrita, todas inerentes ao direito universal à educação e à aprendizagem ao longo da vida, desígnios que a UNESCO nos estimula a concretizar, dia após dia.

Celebrar o dia internacional da educação, assinalado precisamente há um mês, é ir ao encontro desta inovação – curricular, pedagógica, tecnológica – que permita contribuir para atender às necessidades dos nossos alunos, estudantes, professores, formandos, enfim, numa palavra, dos educadores em todo o mundo. Uma inovação contextualizada, holística e transformadora, que permita contribuir para assegurar, aos indivíduos e aos coletivos, o desempenho consciente de um papel ativo, crítico e interventivo na sociedade.

Nos *Saberes em Movimento, Saberes que Movimentam* aqui reunidos, há ainda espaço e tempo para recordar que a *Educação* mudou significativamente, em alguns pontos do globo, mais do que noutros, durante a COVID-19. Esta pandemia, a par de outras situações atuais de grande adversidade, continua a provocar mudanças com impacte no nosso presente e futuro. Importa, pois, (re)imaginar direções positivas para a educação[1]; importa, portanto, (re)imaginar os nossos futuros em conjunto[2]. E que os Saberes plasmados nestes capítulos possam ser o ponto de partida para (re)iniciar esta conversa, tão essencial quanto vital[3]!

Teresa Cardoso

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[1] cf. https://portal.uab.pt/investigacao/projetos/rapide-reimagining-a-positive-direction-for-education/ Acesso em: 25 fev. 2023.

[2] cf. https://unesdoc.unesco.org/ark:/48223/pf0000381115 Acesso em: 25 fev. 2023.

[3] cf. https://www.guninetwork.org/publication/unesco-futures-education-report-reimagining-our-futures-together-new-social-contract Acesso em: 25 fev. 2023.

SUMÁRIO

CAPÍTULO 11

CONSTRUCTION OF ARITHMETIC-ALGEBRAIC THINKING IN A SOCIO-CULTURAL INSTRUCTIONAL APPROACH[1]

Data de submissão: 06/12/2022
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ABSTRACT: We present the results of a research project on arithmetic-algebraic thinking that was carried out jointly by a team in Mexico and another in Quebec[2]. The project deals with the concepts of variable and covariation between variables in the sixth grade at the elementary level and the first, second, and third years of secondary school – namely, children from 11 to 14 years old. We target secondary students (first year or K7) in this text. Our objective relates to the development of a gradual generalization in arithmetic-algebraic thinking in a socio-cultural approach to the learning of mathematics. We experimented with investigative situations using a paper-and-pencil approach and technology. We analyze the emergence, in this context, of a visual abstraction, the production of institutional and non-institutional representations, a sensitivity to contradiction, and, finally, the concepts of variable and of covariation between variables.

KEYWORDS: Gradual generalization. Socio-cultural approach. Arithmetic-algebraic thinking.

1 INTRODUCTION: STEPS OF THE PROJECT

The project presented herein has been ongoing since 2008, carried out jointly by a team in Mexico and another in Quebec. The experimentation was done at the primary and secondary levels as well as in a pre-service teacher education program.

- Step 1: studies of the concept of function (Hitt, González & Morasse, 2008; Hitt & González-Martín, 2015; Hitt & Quiroz, 2019; Passaro, 2009) among students in Secondary 2 and 3 (aged 13-15 year-old, K8 and K9).

[1] This document was published in the proceedings of the PME-NA 42: Hitt, F (2020). Construction of arithmetic-algebraic thinking in a socio-cultural instructional approach. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures*: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico (pp. 120-131). Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

[2] Joint project: Carlos Cortés (UMSNH); Samantha Quiroz (UAC); Fernando Hitt; and Mireille Saboya (UQAM).

- Step 2: a study of the generalization of the concepts of variable and of covariation between variables in relation to arithmetic-algebraic thinking among Secondary 1 students in Quebec (aged 12-13 year-old, K7) (Hitt, Saboya & Cortés, 2017, 2019a, 2019b) and among Secondary 3 students in Mexico.
- Step 3: studies of the concepts of variable and covariation between variables and of the generalization (in the transition from primary to secondary levels) related to arithmetic-algebraic thinking among 6th grade elementary students with learning difficulties in Mexico (11-12 year-old pupils, K6) (Hitt, Saboya & Cortés, 2017a, 2017b; Saboya, Hitt, Quiroz & Antoun, 2019).

Páez's (2004) doctoral thesis worked on teacher training with a teaching method based on collaborative learning, scientific debate, self-reflection, and the process of institutionalization (ACODESA) (see Hitt, 2007).

In order to use the same method in our project, which targeted elementary and secondary students, we had to use, in Step 1 of the project, the results obtained among Secondary 2 and 3 students to create theoretical tools which would allow us to better analyze students' spontaneous representations and their role in the resolution of non-routine situations.

Step 2, which is the focus of this paper, will allow us to better understand the processes of abstraction[3] that trigger a generalization among students (in Secondary 1 in Quebec) in the transition from elementary to secondary levels as well as the construction of a cognitive structure related to arithmetic-algebraic thinking (which we elaborate further below).

We are currently in the process of analyzing the results of Step 3.

2 THEORETICAL FRAMEWORK: SOCIO-CULTURAL APPROACH TO LEARNING

Our approach to the construction of knowledge is based on the notion of activity from Leontiev's (1978) activity theory. According to Leontiev, activity, mediated by mental reflection that situates a subject in the objective world, follows a system of social relations. Leontiev holds that an individual's activity depends on their place in society and their life circumstances (idem, p. 3). Further, activity is intimately related to a motive: "different activities are distinguished by their motives. The concept of activity is necessarily bound up with the concept of motive. There is no such thing as activity without a motive" (idem,

[3] The following is a translation of the definition of abstraction in the Larousse dictionary: an intellectual operation which consists in isolating by thought a characteristic of an object and considering it independently of the other characteristics of that object.

p. 6). Hence, the activity of an individual in a society has a central role in the "subject-activity-object" relation (known as Leontiev's triangle) which, in turn, is part of a system of relations within the given society.

> It stands to reason that the activity of every individual depends on his place in society, on his conditions of life... The activity of people working together is stimulated by its product, which at first directly corresponds to the needs of all participants. (p. 3-6)

Engeström (1987, 1999) analyzes Leontiev's triangle as a model of the relation between subject, object, and artefact-mediation, and concludes that Leontiev's triangle does not capture all elements and relations of a system:

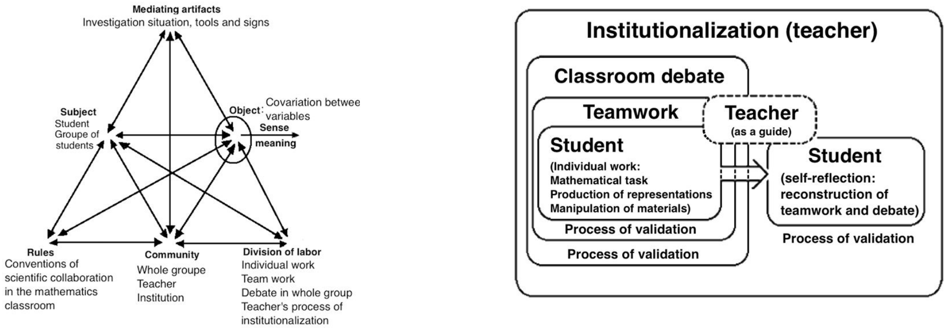
> I am convinced that in order to transcend the oppositions between activity and process, activity and action, and activity and communication, and to take full advantage of the concept of activity in concrete research, we need to create and test models that explicate the components and internal relations of an activity system...To overcome these limitations, the model may be expanded. (p. 29-30).

Voloshinov's (1929/1973) ideas about the construction of sign emphasize the importance of the collaborative work that enriches Leontiev and Engeström's theoretical approach: "[t]he reality of the sign is wholly a matter determined by that communication. After all, the existence of the sign is nothing but the materialization of that communication. Such is the nature of all ideological signs" (p. 13).

Building on the ideas above, we adapted Engelström's model (see Figure 1) while adhering to the ACODESA teaching method (Hitt, 2007). The mathematics classroom is viewed as a microsociety whose various members are the teachers, the students, the institution, and the tools used in the co-construction of knowledge through the resolution of investigative situations (physical materials, school textbooks, computers, etc.).

Collaborative work, communities of practice, and even societies, according to Engeström (1999), Legrand (2001), Leontiev (1978), and Wenger (1998), among others, involve a motive, rules, a division of labour among members, a mediation of artefacts, and interaction among the various actors (see Figure 1). In our case, given a mathematical task, we are interested in the co-construction of students' knowledge through the evolution of their representations in the context of an ACODESA method of teaching.

Figure 1. Engeström's (1999) model adapted to have covariation between variables as object; the various phases of the ACODESA method of teaching.



3 LOCAL THEORETICAL FRAMEWORK AND FIRST ELEMENTS OF ARITHMETIC-ALGEBRAIC THINKING

Given that we are primarily interested in the co-construction of knowledge, we searched for theoretical elements specific to moments of understanding or to the epistemic actions of Pontecorvo & Girardet (1993):

a) Higher-level methodological and metacognitive procedures; and
b) explanation procedures used for the interpretation of particular elements of the task.

To better understand the epistemic actions taking place during the resolution of a mathematical task, we use Rubinshtein's (1958) notions (cited in Davidov, 1990, p. 93-4) about the distinction between "visual empirical thought" and "abstract theoretical thought." In our project, just as in Rubinshtein and his group's, we are interested in the *gradual generalization* that occurs in a collaborative process of learning. For Davidov (idem), generalization is a process: "[i]f we mean the *process* of generalization, then the child's transition from a description of the properties of a particular object to finding and singling them out in a whole class of similar objects is usually indicated" (p. 5).

In the previous century, research about the transition from arithmetic to algebra focused on the concepts of epistemological obstacle (Vergaud, 1988), cuts (Filloy & Rojano, 1989), and gaps (Herscovics & Linchevski, 1994). Today, a change of paradigm purports that cognitive difficulties can be overcome (by a majority of students) with appropriate teaching. The discussion is one of a *continuum* rather than a *rupture* (Hitt, Saboya, and Cortés, 2017a). In this new paradigm, three types of approaches have emerged:

- "Early Algebra," which is based on a functional thinking approach with "an early inclusion of algebraic symbols as a valuable tool for early algebraic thinking" (Carraher, Schliemann, & Brizuela, 2000; Kaput, 1995, among others);

- "Algebraic nature of arithmetic" (Fujii 2003, among others); and
- a "development of algebraic thought" which acts as a support from which to delve deeper into arithmetic (Davidov, 1990; Kilpatrick, 2011; Radford, 2011a, 2011b, among others).

The Early Algebra approach prioritizes the use of institutional algebraic symbols to express covariation between variables and functions (tables of values and algebraic notations of the type $n \rightarrow n + 3$, for example). The second approach is similar to the first, albeit with a broader focus on the use of algebraic symbols in classical arithmetic tasks (see below in Section 3.2). However, the third approach relates to the use of general mathematical notions such as intuition, abstraction, and generalization in a socio-cultural learning of mathematics.

We situate ourselves in this third, socio-cultural type of approach (Engeström 1987, 1999; Leontiev, 1978; Voloshinov, 1929/1973) to the learning of mathematics (Radford's Theory of Objectification, 2011b). We propose the development of complex intuitive ideas by considering, for example, mathematical visualization (including "visual empirical thought" and "abstract theoretical thought," Rubinshtein, 1958), generalization (Davidov, 1990; Radford, 2011a), and the promotion of sensitivity to contradiction (Hitt, 2004) in mathematical activity. We worked on these general notions in elementary and secondary schools; specifically, we worked on the notions of variation and covariation between variables with the aim of developing arithmetic-algebraic thinking in students.

The notion of arithmetic-algebraic thinking is related to the development of a cognitive structure that we wish to promote in students, a structuring structure (a *habitus*) in the sense of Bourdieu (1980): the conditioning associated with a particular class of living conditions produces habitus, systems of durable and transposable dispositions, structured structures predisposed to function as structuring structures (p.88-89).

In our project, we attempt to show how to develop a structuring structure related to arithmetic-algebraic thinking in a mathematics classroom that is viewed as a micro society.

3.1 CO-CONSTRUCTION OF KNOWLEDGE AND A SENSITIVITY TO CONTRADICTION IN THE HISTORY OF MATHEMATICS

Szabó's (1960) studies of the history of mathematics detail elements that, during the Golden Age of the Greek civilization, contributed to the transformation of an empirical-visual mathematics into a definition-based on an axiomatic deductive science. We highlight the following elements:

a) The socio-political progress of the Greeks that allowed for the development of the art of rhetoric, polemical discussion, and critical thinking;
b) the influence of the philosophy of Parmenides of Elea and his disciple, Zeno of Elea (and, in particular, his paradoxes), on the Pythagoreans, who had an interest in mathematics; and
c) a “sensitivity to contradiction” when confronted with mathematical results developed by the Babylonians and the Egyptians, which did not always agree (e.g. the area of the disc).

Indeed, Szabó (idem) shows that Thales of Miletus’ results were obtained in an empirical-visual manner. Szabó (idem) also gives the example of Plato’s (4th century B.C.) Socratic dialog, Meno, which deals with the doubling of the area of a unit square. At the end of the dialog, a slave builds a square on the diagonal of the original unit square. It is easy, visually, to see that the surface area of the new square is double that of the first.

Parmenides’ philosophy on the existence of being excludes non-being and provides the first reflections on logic and on the law of excluded middle. Szabó believes Parmenides influenced the Pythagoreans and that they, in turn, influenced mathematics, creating not only critical thinking but also a sensitivity to contradiction in mathematics. Szabó states:

> The earliest Greek mathematicians, the Pythagoreans, borrowed the method of indirect demonstration from the Eleatic philosophy; consequently, the creation of deductive mathematical science can be attributed to the influence of the Eleatic philosophy. (p. 46)

Unfortunately, many of the Greeks’ documents have been lost. Nevertheless, historians point to Euclid’s Elements, which record the content of the Pythagoreans’ books (Books VII, VIII, IX, and X). In Euclid’s Elements, it is common to find theorems proved by contraposition. Vitrac (2012) confirms that indirect demonstrations (known as reduction to absurdity) are not uncommon in Euclid’s Elements; they appear in a hundred or so propositions (p. 1).

One of Szabó’s main assertions is that the transformation of mathematics into a deductive science (from the 5th century B.C. to the 3rd century B.C.) was accompanied by a transformation of mathematics into an anti-illustrative science. The visual demonstration of the duplication of the surface area of a unit square did not have a place in the new approach in Euclid’s Elements. In Euclid, the illustration did not play a role in the visual demonstration process, but rather as an aid to the formal demonstration.

Historians report that the birth of algebra as a discipline was developed by the Persian al-Khwarizmi (790-850). Hence, while algebra did not originate with the Greeks, they did lay the groundwork for critical thinking, mathematical logic, indirect proof, and a sensitivity to contradiction. This type of thinking is, historically, an important precursor to the development of algebra.

How can we draw inspiration from the history of mathematics in the classroom? How can these historical elements of different cultures be integrated into the mathematics classroom?

3.2 SENSITIVITY TO CONTRADICTION IN THE CONSTRUCTION OF ARITHMETIC-ALGEBRAIC THINKING

Research from the 1980s offers a glimpse into students' difficulties in solving algebraic problems. We consider, as an example, Fujii's (2003) study of the success rates among elementary and high-school students in the United States and in Japan in solving the following two problems:

Problem 1. Mary has the following problem to solve: "Find value(s) for x in the expression: x + x + x = 12" She answered in the following manner. a. 2, 5, 5; b. 10, 1, 1; c. 4, 4, 4 Which of her answer(s) is (are) correct? (Circle the letter(s) that are correct: a, b, c)	Problem 2. Jon has the following problem to solve: "Find value(s) for x and y in the expression: x + y = 16" He answered in the following manner. a. 6, 10; b. 9, 7; c. 8, 8 Which of his answer(s) is (are) correct? (Circle the letter(s) that are correct: a, b, c) State the reason for your selection.

> It is also important to note that it is rare for students to get both problems correct, which was also consistent with the data for both countries [USA and Japan]. Let me select the Athens (GA) 6th, 8th and 9th graders from the American data, simply because these students have a common educational environment. The percentages of correct answers for 6th, 8th, and 9th grade are 11.5%, 11.5% and 5.7% respectively. For Japanese students, the correct response from 5th, 6th, 7th, 8th, 10th and 11th grades are 0%, 3.7%, 9.5%, 10.8%, 18.1% and 24.8% respectively (Fujii, 1993).

These problems help distinguish between students with a *conception* of the role of a variable in an algebraic expression and those who had formed the *concept* of a variable.

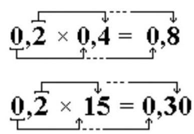
By analyzing the tasks Fujii (2003) proposes, we see they had been designed as assessment tools (to detect the conceptions students had formed). The design of a task meant to promote learning based on students' conceptions, however, is a whole other matter. In what follows, we present two examples of sensitivity to contradiction.

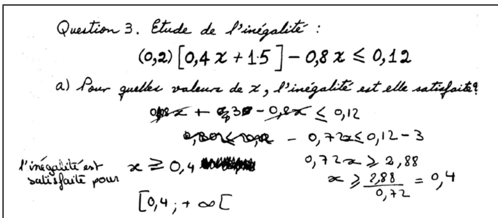
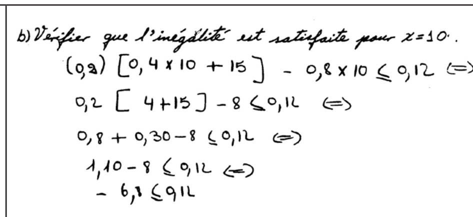
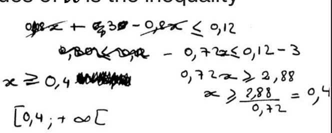
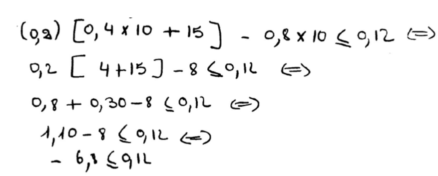
3.2.1 First example

Sensitivity to contradiction in the process of solving the following:

a) Solve this inequality: *0.2(0.4x + 15) – 0.8x ≤ 0.12*
b) Verify that $x = 10$ is an element of the solution set.

In designing this activity, we took into account Brousseau's (1997) notion of epistemological obstacle in the learning of decimal numbers: an error that results when knowledge that, in other situations, had been valid and effective proves to be erroneous in a new situation. In this case, an error occurs when knowledge about multiplication of natural numbers is applied to multiplication of decimal numbers. We take advantage of this error to promote a richer mathematical structure: a sensitivity to contradiction. Here is an example of a student's work:



	
Question 3. Study of the inequality: $(0.2)[0.4x + 1.5] - 0.8x \leq 0.12$ a) For which values of x is the inequality satisfied? The inequality is satisfied for 	b) Verify that the inequality is satisfied for $x = 10$: 

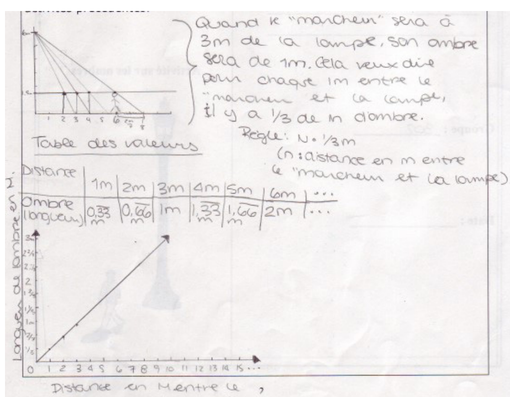
We note the student made the mistakes anticipated by the researcher. The student had proposed the solution "st [solution] = 0,12," but after addressing question b), the student noticed the contradiction. The student retraced his steps to resolve the contradiction in part a). He spotted and overcame the cognitive contradiction, even if, formally, the contradiction remained in item b). This shows the student is sensitive to contradiction.

3.2.2 Second example

The *Shadow Situation* was one of five situations proposed in a month-and-a-half-long experiment with students in their third year of high-school. The five (sequential) situations were worked on in connection with the ACODESA method and with the goal of developing the concepts of covariation between variables and of function (Hitt & González-Martín, 2015; Hitt & Morasse, 2009). The following is a translation of the Shadow Situation given to students:

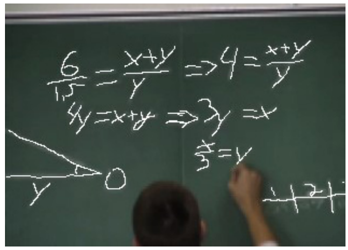
> Suppose we have a source of light with a height of 6 meters (a streetlight). We consider the shadow formed when a when a person who is 1.5 meters tall walks down the street. We are interested in the relationships between the quantities involved.
>
> 
>
> *Are some of the quantities dependent on one another? Which ones?*
>
> Select two quantities that depend on one another and describe the phenomenon with the various representations you used in previous activities.

Phase 1: Individual work. Two girls work on their own[4] to understand the task. One of them represented the situation through a proportional drawing. Starting with an empirical-visual thought, she found a relationship between the quantities "distance travelled by the person" and "length of the shadow."



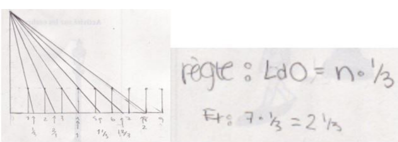
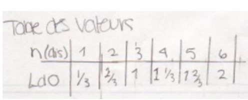
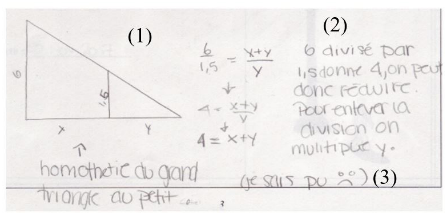
Phase 2: Teamwork (Prusak, Hershkowits, & Schwarts, 2013, suggest groups of two to three). The two girls produce a verbal description of a relationship, an algebraic expression, and a graphical representation of the situation.

[4] Translation of text in top right corner of image: [w]hen the "walker" will be at 3m from the lamp, his shadow will be 1m. This means that for every 1m between the "walker [*sic*] and the lamp, there is 1/3 of a m of shadow. Rule: N * 1/3m (n [*sic*]: distance in m between the "walker [*sic*] and the lamp). Table of values, first row: distance; table of values, second row: shadow (length). Graph, x-axis: distance in M [*sic*] between [*sic*]; graph, y-axis: length of the shadow in M [*sic*].



Phase 3: Classroom debate. One group of students failed to find an answer due to algebraic errors. Upon seeing the two girls' results, this group manages to construct an algebraic approach by using similar triangles.

Phase 4: Self-reflection. The instructor collects everything the students produced and re-assigns them the situation as homework, this time with the instruction to re-create the work done in class. The following is what one of the girls (mentioned above) produced as a reconstruction of what had been discussed in class:

C's reconstruction of the work done with her team	C's reconstruction of the classroom debate
 	

She reconstructed with no difficulty what she had done numerically and visually with her team-mate. Unfortunately, when she wanted to reconstruct the boys' algebraic process, she made a mistake and failed to come up with a solution. In her drawing (the one on the right), she expressed a feeling of unease in the face of a contradiction she couldn't overcome. This shows she had formed a sensitivity to contradiction. From a cognitive standpoint, *a sensitivity to contradiction is an awareness of contradiction accompanied by a sense of unease, and its resolution by a sense of happiness.*

These examples demonstrate the importance of students' spontaneous representations. Given these findings on students' spontaneous representations, Hitt and Quiroz (2019) proposed the notion of ***socially-constructed representation***, one which materializes through the evolution of students' functional-spontaneous representation as it emerges in individual work and is then discussed in a team, in large groups, and in self-reflective work. According to Hitt and Quiroz (2019, p. 79),

> [a] socially-constructed representation is one that emerges in individuals when given a non-routine activity; the actions in the interaction with the situation have functional (mental, oral, kinesthetic, schematic) characteristics and are related to a spontaneous (external) representation. The representation is functional in the sense that the student needs to make sense of the situation, and it is spontaneous because it naturally occurs in an attempt to understand and solve the non-routine situation. [Translation]

4 THE INVESTIGATIVE SITUATION (THE TASK): KEY ELEMENT IN THE CO-CONSTRUCTION OF MATHEMATICAL KNOWLEDGE

The theories of didactical situations (Brousseau, 1998), of "problem solving" (Mason, Burton, & Stacey, 1982; Schoenfeld, 1985), and of Realistic Mathematical Education from Freudenthal (1991) have prompted changes in curricula worldwide. There is a break from the classical approach – that is, from "definition-theorem-exercises and problems" instruction. Situational problems, problems in general, and contextualized problems have a fundamental role to play in the new approach. In light of these theories, task design is viewed as central for overcoming cognitive barriers. A new era came for the organization and role of the task in mathematics instruction, in situational problems related to creativity, and in mathematical modelling (Blum, Galbraith, Henn, & Niss, 2007; Hitt & González-Martín, 2015; Hitt, Saboya, & Cortés, 2017; Hitt & Quiroz 2019; Lesh & Zawojewski, 2007; Margolinas, 2013).

The activities we designed are related to the ACODESA teaching method in a socio-cultural approach to mathematics instruction. We call our activities "investigative situations":

> An **investigative situation** consists of different tasks that follow the steps of the ACODESA method. The tasks attempt to promote, first and foremost, the emergence of non-institutional or institutional representations, empirical-visual thinking related to diversified thinking (that is, divergent thinking), conjecture, prediction, and validation. In second and third stages (teamwork and classroom debates), we try to promote abstract thinking that includes sensitivity to contradiction as well as an evolved version of the representations and characteristics formed in the first stage. In a fourth stage, students re-construct what had been done in class so as to solidify the knowledge they had formed. Finally, the teacher reviews students' various solutions and presents the institutional position vis-à-vis the content considered in the situation. [Translation]

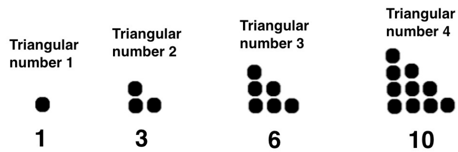
The design of investigative situations follows an organization such as that outlined in Hitt, Saboya, and Cortés (2017b).

4.1 VARIATION AND COVARIATION BETWEEN VARIABLES: AN EXAMPLE WITH POLYGONAL NUMBERS

We now present the first step of an investigative situation that involves polygonal numbers, and which is targeted towards students in their first year of high-school. This step consisted of five questions to be solved with paper and pencil. The second step had students use technology to validate their conjectures. In total, the situation was eight pages long. The following is a translation from French:

Step 1 (Individual work, followed by teamwork; paper-and-pencil approach)

A long, long, long, long time ago (around 520 B.C.), a mathematician called Pythagoras founded a school on an island in ancient Greece. He and his students were fascinated by both numbers and geometry. One of their ideas consisted of representing numbers by geometric figures. They called these *polygonal numbers*. For example, they noticed that certain numbers could be represented by triangles. Thus, 1, 3, 6, and 10 are the first four triangular numbers since they can be represented by points arranged in triangles as follows:



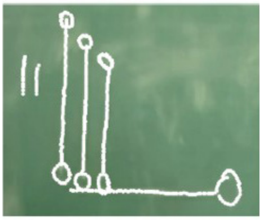
1) Observe these numbers carefully. What is the fifth triangular number? Represent it. Explain how you did this.
2) How do you think a triangular number is constructed? What do you observe?
3) What is the 11th triangular number? Explain how you found its value.
4) You must write a SHORT email to a friend describing how to calculate the triangular number 83. Describe what you would write. YOU DON'T HAVE TO DO ANY CALCULATIONS!
5) And how would you calculate any triangular number? (We want a SHORT message here as well.)

4.1.1 Teams' Responses to Questions 1, 2, and 3

In this first step, we wanted to promote empirical-visual thinking (Rubinshstein, 1973) and generalization (Davidov, 1990; Radford, 2011). Students (in teams G1 and G3) naturally shifted from a visual approach to an arithmetic procedure (an epistemic action). For example, to calculate T_{11}, they wrote 1+2+3+4+5+6+7+8+9+10+11.

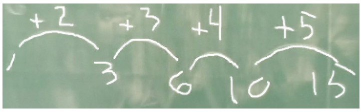
1+2+3+4+5+6+7+8+9+10+11

Team G2 first moved from a concrete visual approach to a more general visual approach and then to an arithmetic procedure (an epistemic action). Hence, for T_{11}, they wrote 11+10+9+8+7+6+5+4+3+2+1.



11+10+9+8+7+5+4+3+2+1

We note the abandonment of the iconic representation by one student (from team G4) who, during the classroom debate, switched from a detached visual approach to the polygonal configurations to an iterative calculation which he had not discussed with his teammates (what Rubinshtein would term theoretical abstract thinking). Team G4 used this final strategy, along with Excel, to tackle the fifth question of the second step of the investigative situation.



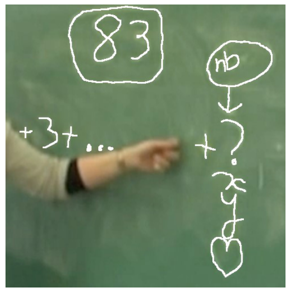
This shift shows the importance of teamwork and of Yan's reflection as he organized his thoughts (in what Vygotsky, 1932/1962, would call inner speech) so as to communicate them to the group (Voloshinov's construction of sign). Yan needed to make himself understood by the rest of the class.

4.1.2 Team Responses to Questions 4 and 5 and First Classroom Debate

The following are the responses given by each team in the first classroom debate:

	Team's response to question 4	Team's response to question 5
G1	We add all the numbers from 1+2+3... all the way to the number of points on the side.	We add all the numbers from 1+2+3... all the way to the number of points on the side.
G2	Add up the numbers from 1 to 83.	You add up the numbers from there?
G3	You have to do 83+82+81... all the way to 1.	Calculate the last diagonal column and calculate by doing -1 to the number. E.g. 15th, 15+14+13... etc.
G4	You have to do; 1+2+3+4+5+6+7+8... +83 and this will give you the answer.	You put the same number on the other sides and then you add up 1+2+3+4+5+6... until you get to your number and your answer is the triangular number.

During the classroom debate, the researcher asked what answers had been written in response to question 5 (see responses above), which asked for a short message describing how to find any triangular number. The students first suggested the sum “1+2+3 all the way to your number.” The researcher intervened: how can I write a number I don’t know? Different proposals emerged. The first was to write “?” ; afterwards, they proposed “x” or “y.” The teacher asked whether a heart could be used: “♥.” One student replied that they could use anything that wasn’t a number.



The students transitioned from empirical-visual thinking to abstract arithmetic-algebraic thought. The variable was first expressed in words: “all the way to your number.” Then, it was expressed as “?,” then, as “x” or “y,” and, finally: “we can use anything that isn’t a number.”

Output produced by team G4 (during teamwork and during the classroom debate)			
Teamwork and spontaneous generalization	Surprise at finding a decimal number as T_{100} and team discussion towards generalization (with Excel)	General computation of a triangular number presented to the rest of the class	Generalization obtained through classroom debate
(83+1)÷2=42 42x83 =	100 (100+1)÷2=X X x 100 =	(46+1)÷2 = 47 ÷ 2 =23,5 46x23,5=(1081)	(x+1)÷2=y y·X=

We note that each abstraction came with a certain type of generalization. The processes of abstraction were of the following types: *visual abstraction, arithmetic abstraction, emergence of the concept of a variable, emergence of the concept of covariation between variables*.

4.1.3 Forty-five Days Later: Phase of Self-Reflection (Reconstruction)

During this step, Yan, the student who found an algebraic expression for triangular numbers, tried to remember his formula but got it wrong. He wrote: ***(Row*2)-1=y*** and

(Row*y=triangular number. During the pentagonal number activity which we had given him as a challenge, he wrote that ***Row*(Row + (Row * 0.5 – 0.5)) = pentagonal number***. He found this expression by using the same strategy he had used 45 days beforehand to deal with triangular numbers. This expression is equivalent to the institutional one: $P_n = \frac{n(3n-1)}{2}$.

During this process of self-reflection, another student obtained the following in response to the question about triangular numbers: ***Odd number: (row + 1) ÷ 2 * row = triangular number.*** This expression is equivalent to (when restricted to odd numbers).

5 CONCLUSIONS

In this paper, we wanted to show the various elements needed for the construction of arithmetic-algebraic thinking. Building on a few ideas from the history of mathematics, from a socio-cultural theory of learning, and from the ACODESA teaching method, we have shown that for the construction of arithmetic-algebraic thinking, various elements of the mathematics classroom need to be taken into account: the role of the task (investigative situations) in the acquisition of knowledge, communication in the classroom, mathematical visualization, the role of non-institutional and institutional representations, generalization, conjecture, sensitivity to contradiction, validation, and proof.

Our approach seeks to develop and enrich an association between arithmetic and algebra (*a habitus*) to promote the construction of a structuring structure, in the sense of Bourdieu (1980), that is related to arithmetic-algebraic thinking and which supports not only algebra, but also an enrichment of the cognitive structure of arithmetic tasks.

Further, we observed the emergence of the concepts of variable and of covariation between variables through the process of co-construction of knowledge.

The results of our studies have encouraged us to experiment new investigative situations (following the ACODESA method) in grade 6 classrooms. So far, we have suggested five investigative situations of different types and which require electronic tablets: Marcel's Restaurant, The El Dorado Jewelry Shop, Windows, The Garden and the Pumpkins, and Rectangles and Disks. We are currently analyzing the results.

REFERENCES

Blum, W., Galbraith, P., Henn, H. & Niss, M. (Eds. 2007). *Modelling and applications in mathematics education*. The 14th ICMI Study. New York: Springer.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. 1970-1990, In Balacheff, N., Cooper, M., Sutherland, R. And Warfield, V. (Eds. and Trans.) Dordrecht: Kluwer.

Bourdieu, P. (1980). *Le sens pratique*. Paris : Éditions de Minuit.

Carraher, D., Schliemann, A., & Brizuela, B. M. (2000). Early algebra, early arithmetic: Treating Operations as Functions. Annex to the PME-NA XXII proceedings (pp. 1-24), Tucson, Arizona, USA.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37*(2), 87-115.

Davydov, V. V. (1990). Types of Generalization in Instruction: Logical and Psychological Problems in the Structuring of School Curricula. Reston, VA: National Council of Teachers of Mathematics.

Engeström, Y. (1987). Learning by expanding: An activity-theoretical approach to developmental research. Helsinki: Orienta-Konsultit.

Engeström, Y. (1999). Activity theory and individual and social transformation. In Y. Engeström, R. Miettinen & R.-L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 19-38). New York, NY: Cambridge University Press.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. For the Learning of Mathematics, 9(2), 19–26.

Freudenthal, H. (1991). *Revisiting Mathematics Education*. Dordrecht: Kluwer.

Fujii, T. (2003). Probing students' understanding of variables through cognitive conflic problems: Is the concept of a variable so diff cult for students to understand? In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education (PME) (Vol. 1, pp. 49–65).

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. Educational Studies in Mathematics, 27(1), 59–78.

Hitt, F. (2004). Les représentations sémiotiques dans l'apprentissage de concepts mathématiques et leur rôle dans une démarche heuristique. En Gisèle Lemoyne (Ed.), *Le langage dans l'enseignement et l'apprentissage des mathématiques: complexité et diversité des cadres d'étude. Revue des Sciences de l'Éducation*. Volume XXX, no. 2, pp. 329-354.

Hitt F. (2007). Utilisation de calculatrices symboliques dans le cadre d'une méthode d'apprentissage collaboratif, de débat scientifique et d'auto-réflexion. In M. Baron, D. Guin et L. Trouche (Éds.), *Environnements informatisés et ressources numériques pour l'apprentissage. Conception et usages, regards croisés* (pp. 65-88). Paris: Hermès.

Hitt, F., Gonzalez, A. & Morasse, C. (2008). Visualization and students' functional representations in the construction of mathematical concepts. An example: The concept of co-variation as a prelude to the concept of function. In 11th International Congress on Mathematics Education (ICME-11), Topic Study Group 20 (TSG 20), Visualization in the Teaching and Learning of Mathematics, July 6-13, 2008, Monterrey, N. L., Mexico. http://tsg.icme11.org/tsg/show/21.

Hitt, F. & González-Martín, A.S. (2015). Covariation between variables in a modelling process: The ACODESA (Collaborative learning, Scientific debate and Self-reflexion) method. *Educational Studies in Mathematics*, *88*(2), 201-219.

Hitt, F. et Morasse, C. (2009). Développement du concept de covariation et de fonction en 3ème secondaire dans un contexte de modélisation mathématique et de résolution de situations problèmes**.** Proceedings CIEAEM 61. *Quaderni di Ricerca in Didattica Matematica,* G.R.I.M. Retrived from http://math.unipa.it/~grim/cieaem/quaderno19_suppl_2.htm

Hitt, F. et Quiroz, S. (2019). Formation et évolution des représentations fonctionnelles-spontanées à travers un apprentissage socioculturel. *Annales de didactique et de Sciences Cognitives*, vol. 24, 75-106.

Hitt, F., Saboya, M. and Cortés C. (2017a). Rupture or continuity: the arithmetico-algebraic thinking as an alternative in a modelling process in a paper and pencil and technology environment. *Educational Studies in Mathematics*, *94*(1), 97-116.

Hitt, F., Saboya, M. and Cortés C. (2017b). Task design in a paper and pencil and technological environment to promote inclusive learning: An example with polygonal numbers. In G. Aldon, F. Hitt, L. Bazzini & Gellert U. (Eds.), *Mathematics and technology. A C.I.E.A.E.M. Sourcebook* (pp. 57-74). Cham: Springer.

Kaput, J. (1995). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. Paper presented at the Annual Meeting of the National Council of Teachers of Mathematics, Boston, MA.

Legrand, M. (2001). Scientific debate in mathematics courses. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI Study,* 127-135, Dordretch: Kluwer Academic Publishers.

Leontiev, A. (1978). Activity, counciousness, and personality. Englewood Cliffs, NJ: Prentice Hall.

Lesh, R. & Zawojewski, J. S. (2007). Problem solving and modeling. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763-804). Greenwich, CT: Information Age Publishing.

Margolinas, C. (Ed.) (2013). Task design in mathematics education. Proceedings of ICMI Study 22. Jul 2013, Oxford, United Kingdom. https://hal.archives-ouvertes.fr/hal-00834054v3/document.

Mason, J., Burton, L., & Stacey, K. (1982). Thinking mathematically. Wokingham: Addison Wesley.

Páez, R. (2004). Procesos de construcción del concepto de límite en un ambiente de aprendizaje cooperativo, debate científico y autorreflexión. Unpublished thesis. Cinvestav-IPN. México.

Passaro, V. (2009). Obstacles à l'acquisition du concept de covariation et l'introduction de la représentation graphique en deuxième secondaire. *Annales de Didactique et des Sciences Cognitives*, *14*, 61-77.

Pontecorvo, C., & Girardet, H. (1993). Arguing and reasoning in understanding historical topics. *Cognition and Instruction, 11*, 365-395.

Prusak, N., Hershkowits R. & Schwarz B. (2013). Conceptual learning in a principled design problem solving environment. *Research in Mathematics Education*, 15(3), pp. 266-285.

Radford, L. (2011a). Grade 2 students' non – symbolic algebraic thinking. In J. Cai & E. Knuth (eds.), *Early Algebrization, Advances in Mathematics Education* (pp. 303-322). Dordrecht: Kluwer.

Radford, L. (2011b). Vers une théorie socio-culturelle de l'enseignement-apprentissage: la théorie de l'objectivation. *Elements 1*(1), 1-27. Toulouse: IREM.

Rubinshtein, S. L. (1958). 0 *myshlenii iputyakh ego issledovaniya* [*On Thought and Ways of Investigating It*]. Moscow: Publishing House of the USSR Academy of Sciences.

Saboya, M., Hitt, F., Quiroz S. et Antun Z (en presse). La pensée arithmético-algébrique comme transition du primaire au secondaire: des situations d'investigation dans lesquelles modélisation et technologie jouent un rôle central. *Actes de l'EMF 2018*, Paris.

Schoenfeld, A. (1985). *Mathematical Problem Solving*. New York: Academic Press.

Szabó, Á. (1960). The transformation of mathematics into deductive science and the beginnings of its foundation on definitions and axioms. *Scripta Mathematica*, *XXVII* (I), 27-49, (II), 113-139.

Vergnaud, G. (1988). Long terme et court terme dans l'apprentissage de l'algèbre. In C. Laborde (Ed.), Actes du Premier Colloque Franco-Allemand de Didactique des Mathématiques et de l'informatique (pp. 189–199). Grenoble: La Pensée Sauvage.

Vitrac, B. (2012). Les démonstrations par l'absurde dans les Éléments d'Euclide : inventaire, formulation, usages. Séminaire d'épistémologie et d'histoire des idées mathématiques. Paris, France. hal-00496748v2.

Voloshinov, V. N. (1973). *Marxism and the phylosophy of language*. Translated by Matejka L. And Titunik I. R. Cambridge, MA: Harvard University Press.

Vygostky, L. (1932/1962). *Thought and Language.* Cambridge, Mass.: MIT Press.

Wenger, E. (1998). *Communities of practice. Learning, meaning, and identity*. Cambridge: Cambridge University Press.

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ÍNDICE REMISSIVO

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B

C

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